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The quantised MIC–Kepler problem and its symmetry group for negative energies

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Abstract. The complex line bundles L_m over $\boldsymbol{R}^3-\{0\}$, which are associated with the principal S^1 bundle $\pi: \boldsymbol{R}^4-\{0\}\to\boldsymbol{R}^3-\{0\}$, the m being integers, are introduced in order to reduce the quantised four-dimensional conformal Kepler problem by the S^1 action. For any m, the reduced quantum system, which is defined on the space of L^2 cross sections in L_m, describes the quantised three-dimensional Kepler problem (hydrogen atom) along with a centrifugal potential and Dirac's monopole field, which is referred to as the quantised MIC-Kepler problem. The numbers $-m/2$ are equal to the eigenvalues of the angular momentum operator corresponding to the S^1 action, and are also interpreted as the strength of the monopole field. The symmetry group of the reduced system, (i) $\mathrm{SU}(2)\times\mathrm{SU}(2)$ (m odd) or (ii) SO(4) (m even), is unitarily irreducibly represented on each negative-energy eigenspace. These representations exhaust all the unitary irreducible representations of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ or SO(4), up to equivalence; a pair of integers (m, n) satisfying the conditions that (i) $|m|\leq n$, $n=0, 1, \ldots$, and (ii) m and n are simultaneously even or odd designate the representations, together with the complex line bundles L_m and the negative eigenvalues E_n. The generators of the symmetry group are shown to be the total angular momentum operator and the Runge-Lenz-like operator. The hydrogen atom ($m=0$) is thus generalised to the quantised MIC-Kepler problem along with the symmetry group.

1. Introduction

In a previous paper (Iwai and Uwano 1986), the four-dimensional conformal Kepler problem was reduced to the three-dimensional Kepler problem with a centrifugal potential and Dirac's monopole field, which is referred to as the MIC-Kepler problem in view of the paper by McIntosh and Cisneros (1970). This dynamical system was also treated by Schonfeld (1980) under the name of the 'q-Kepler problem'. A quick review of this reduction is as follows.

Let $\dot{\boldsymbol{R}}^4 := \boldsymbol{R}^4-\{0\}$ be endowed with the conformally flat metric $\mathrm{d}s_c^2=4r\sum_{j=1}^{j=4}\mathrm{d}x_j^2$, where (x_j) are the cartesian coordinates with $r=\sum_{j=1}^{j=4}x_j^2$, and $T^*\dot{\boldsymbol{R}}^4$ denotes the cotangent bundle of $\dot{\boldsymbol{R}}^4$. The standard symplectic 2-form $\mathrm{d}\theta$ is defined on $T^*\dot{\boldsymbol{R}}^4$ by $\mathrm{d}\theta=\sum_{j=1}^{j=4}\mathrm{d}p_j\wedge\mathrm{d}x_j$, where (p_j) are the conjugate momentum variables with respect to $\mathrm{d}s_c^2$. Then the four-dimensional conformal Kepler problem (Iwai 1981) is defined on the symplectic manifold $(T^*\dot{\boldsymbol{R}}^4, \mathrm{d}\theta)$, together with the Hamiltonian given by

$$H=\frac{1}{2}\left(\frac{1}{4r}\sum_{j=1}^{4}p_j^2\right)-\frac{k}{r}\qquad (k>0,\ \text{constant}).$$

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By an S^1 action on $T^*\dot{\boldsymbol{R}}^4$, the conformal Kepler problem $(T^*\dot{R}^4, \mathrm{d}\theta, H)$ is reduced to the Hamiltonian system $(T^*\dot{\boldsymbol{R}}^3, \sigma_\mu, H_\mu)$. Here, $T^*\dot{\boldsymbol{R}}^3$ is the cotangent bundle of $\dot{\boldsymbol{R}}^3 := \boldsymbol{R}^3 - \{0\}$ and σ_μ is the reduced symplectic form expressed as $\mathrm{d}\theta' + \Omega_\mu$, where θ' is the canonical 1-form on $T^*\dot{\boldsymbol{R}}^3$ and Ω_μ is a 2-form on $\dot{\boldsymbol{R}}^3$ describing Dirac's monopole field of strength $-\mu$. The reduced Hamiltonian H_μ is written as

$$H_\mu = \frac{1}{2}\sum_{j=1}^{3}\tilde{p}_j^2 + \frac{\mu^2}{2r^2} - \frac{k}{r} \tag{1}$$

where $(\tilde{x}_j)$ are the cartesian coordinates in $\dot{\boldsymbol{R}}^3$ with $r^2 = \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2$ and $(\tilde{p}_j)$ are the standard conjugate momentum variables. It should be pointed out that the 2-form Ω_μ is the outcome of the non-zero angular momentum associated with the S^1 action on $T^*\dot{\boldsymbol{R}}^4$, and that the appearance of Ω_μ is attributed to Kummer (1981). The point of this reduction process is that the configuration space $\dot{\boldsymbol{R}}^4$ is made into a principle fibre bundle with structure group $S^1 \simeq \mathrm{U}(1)$; the free S^1 action on $\dot{\boldsymbol{R}}^4$ gives rise to the projection $\dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$. Moreover that action lifts symplectically on $T^*\dot{\boldsymbol{R}}^4$. Note also that the S^1 bundle $\dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$ is contractible to the Hopf bundle $S^3 \to S^2$.

One of the aims of this paper is to construct the quantum system associated with the MIC-Kepler problem $(T^*\dot{\boldsymbol{R}}^3, \sigma_\mu, H_\mu)$. One way to achieve this purpose is to reduce the quantised conformal Kepler problem by the S^1 action. The resultant system will then be considered as the quantum system associated with the MIC-Kepler problem. Here, the quantised conformal Kepler problem is defined by a pair $(L^2(\dot{\boldsymbol{R}}^4; 4r\,\mathrm{d}x), \hat{H})$ (see Iwai 1982a), where $L^2(\dot{\boldsymbol{R}}^4; 4r\,\mathrm{d}x)$ is the space of square integrable complex-valued functions on $\dot{\boldsymbol{R}}^4$ with respect to the volume element $4r\,\mathrm{d}x = 4r\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4$. The $\hat{H}$ is the Hamiltonian operator given by

$$\hat{H} = -\frac{1}{2}\left(\frac{1}{4r}\sum_{j=1}^{4}\frac{\partial^2}{\partial x_j^2}\right) - \frac{k}{r}.$$

The quantised conformal Kepler problem was already reduced to the hydrogen atom of three dimensions (Iwai 1982a), when the momentum eigenvalue of the S^1 action is zero. The reduction process to be carried out in this paper has a deep relation to the work of Greub and Petry (1975) who treated the Hamiltonian operator for the motion of a particle in Dirac's monopole field (see also Ryder 1980, Crampin 1981). In fact, complex line bundles over $\dot{\boldsymbol{R}}^3$ will be introduced to describe the quantised MIC-Kepler problem, which carries Dirac's monopole field.

The main interest centres on the 'quantisation' of the symmetry group SO(4) for the classical MIC-Kepler problem of negative energy. The reduction principle is applied to the symmetry group for the quantised conformal Kepler problem in order to obtain the symmetry group for the quantised MIC-Kepler problem. The connection between the four-dimensional harmonic oscillator and the three-dimensional hydrogen atom is now widely recognised, and hence the symmetry of the hydrogen atom is discussed from this point of view (see Chen and Kibler (1985) and references therein). However, those papers discuss the symmetry at Lie algebraic level. The present paper discusses it from the Lie group point of view on the basis of the complex line bundles over $\dot{\boldsymbol{R}}^3$. It is shown that all the unitary irreducible representations of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ or SO(4) are realised, up to equivalence, in the negative-energy eigenspaces for the quantised MIC-Kepler problem. The infinitesimal generators are also given which act on cross sections in the complex line bundle over $\dot{\boldsymbol{R}}^3$. The contents of this paper are outlined in the following way.

Section 2 is a review in which the quantised conformal Kepler problem is related to the harmonic oscillator to get the negative-energy eigenvalues $E_n = -2k^2/(n+2)^2$, $n = 0, 1, 2, \ldots$.

Section 3 presents the reduction of $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ by the S^1 action. A geometric setting for this reduction process is given by complex line bundles associated with the principal U(1) bundle $\pi: \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$, which were treated in Greub and Petry (1975) to globally describe Dirac's monopole. Fix an arbitrary integer m and consider the unitary irreducible representation ρ_m of $\mathrm{U}(1) \simeq S^1$ on $\boldsymbol{C}$; $z \to \exp(\mathrm{i}mt/2)z$, $z \in \boldsymbol{C}$. Then, the complex line bundle $L_m = (\dot{\boldsymbol{R}}^4 \times_m \boldsymbol{C}, \pi_m, \dot{\boldsymbol{R}}^3)$, $\pi_m: \dot{\boldsymbol{R}}^4 \times_m \boldsymbol{C} \to \dot{\boldsymbol{R}}^3$, is associated with the U(1) bundle $\pi: \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$ (see Greub and Petry 1975, Kobayashi and Nomizu 1963). Note that in Iwai and Uwano (1986) U(1) has the period 4π. The restriction of $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ to the ρ_m-equivariant functions (see Greub and Petry 1975) can be identified with the space of square integrable cross sections in L_m, denoted by Γ_m. The reduced quantum system will be defined on Γ_m. The restriction to the ρ_m-equivariant functions and the introduction of the complex line bundle L_m are the geometric consequence of the conservation of the angular momentum associated with the action of $\mathrm{U}(1) \simeq S^1$, because the ρ_m-equivariant functions are the very eigenfunctions of that angular momentum operator. Thus the introduction of L_m may be interpreted as the reduction process by the S^1 action. The line bundle L_m is known to be endowed with a linear connection which arises from the connection defined naturally in the U(1) bundle $\pi: \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$ (cf Greub and Petry 1975, Kobayashi and Nomizu 1963). Its curvature will define a 2-form on $\dot{\boldsymbol{R}}^3$, which is Dirac's monopole field of quantised strength. The linear connection will be used to describe the reduced Hamiltonian operator in the next section. Note that when m equals zero, the line bundle L_0 is trivial and hence Γ_0 becomes $L^2(\boldsymbol{R}^3)$. This is the case discussed in Iwai (1982a). Throughout the following sections, the results for $m = 0$ give the corresponding results of Iwai (1982a).

Section 4 deals with the reduction of the quantised conformal Kepler problem to give the quantised MIC-Kepler problem. Since the Hamiltonian $\hat{H}$ of the quantised conformal Kepler problem is invariant under the S^1 action, the $\hat{H}$ is reduced to the operator $\hat{H}_m$ acting on (a dense domain in) Γ_m:

$$\hat{H}_m = -\frac{1}{2}\sum_{j=1}^{3} \nabla_j^2 + \frac{(-m/2)^2}{2r^2} - \frac{k}{r} \tag{2}$$

where ∇_j stands for $\nabla_{\partial/\partial \tilde{x}_j}$, the covariant derivation with respect to the linear connection. The $\hat{H}_m$ is interpreted as the quantised Hamiltonian operator of the classical reduced Hamiltonian H_μ in the presence of Dirac's monopole field, because, by the comparison of (1) and (2), one observes that $\tilde{p}_j$ are replaced by $-\mathrm{i}\nabla_j$ and μ is quantised to be $-m/2$. The reduced system $(\Gamma_m, \hat{H}_m)$ will be referred to as the quantised MIC-Kepler problem in view of the work of McIntosh and Cisneros (1970).

In § 5, the eigenspaces of $\hat{H}_m$ for negative eigenvalues are discussed. From the definition of the reduced Hamiltonian operator $\hat{H}_m$ the eigenvalues of $\hat{H}_m$ come from those of $\hat{H}$. The eigenspaces for $\hat{H}_m$ of negative energy are shown to be reduced from the eigenspaces of $(L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x), \hat{H})$ with negative eigenvalues.

In § 6, the symmetry group of $(\Gamma_m, \hat{H}_m)$ is obtained. By making full use of the reduction in § 5 and of the results in the papers of Iwai (1982a, b), the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$, which leaves the space of ρ_m-equivariant eigenfunctions of $\hat{H}$ invariant, can be taken as the symmetry group of the reduced system. To be precise, depending on the parity of m, the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ has the unitary irreducible representation in the eigenspaces for $\hat{H}_m$ of negative energy; if m is odd, $SU(2)\times SU(2)$ is represented, and if m is even, $SO(4)\simeq SU(2)\times SU(2)/\boldsymbol{Z}_2$ is. The pairs of integers m and n, which assign the bundle L_m and the eigenvalue E_n respectively, designate all the unitary irreducible representations of $SU(2)\times SU(2)$ or $SO(4)$, up to equivalence, if (i) $|m|\leq n$, $n=0,1,2,\ldots$, and (ii) m and n are simultaneously even or odd.

In § 7, the generators of the symmetry group of the quantised MIC-Kepler problem $(\Gamma_m, \hat{H}_m)$ are studied. The reduction method for $\hat{H}$ in § 4 is again applicable. The generators of $SU(2)\times SU(2)$ found in Iwai (1982a) for the conformal Kepler problem are used to obtain the generators on Γ_m by the reduction method. Like the classical constants of motions for the reduced system (see Iwai and Uwano 1986), these generators contain the total angular momentum operator and the Runge–Lenz-like operator.

Section 8 contains concluding remarks. In order to see explicitly how the monopole field arises from the S^1 action in $\dot{\boldsymbol{R}}^4$, the conformal Kepler Hamiltonian operator, $\hat{H}$, is expressed in the curvilinear coordinates in $\dot{\boldsymbol{R}}^4$ which are related to the Euler angles. In addition, Iwai (1987) for the quantum planar three-body problem is briefly referred to, which shares the reduction principle with the present paper. Further, a relation to the setting for the monopole harmonics is pointed out. The reduction process in § 3 provides the setting for the monopole harmonics (Dray 1986), if $\dot{\boldsymbol{R}}^4$ and $\dot{\boldsymbol{R}}^3$ are contracted to S^3 and S^2, respectively. In fact, the complex line bundle L_m over $\dot{\boldsymbol{R}}^3$ then contracts to the complex line bundle over S^2. According to Wu and Yang (1976) and Dray (1986), the monopole harmonics are defined as the eigen cross sections in the complex line bundles over S^2 for the operator that are reduced from the standard Laplacian on S^3. This reduced operator equals, within an additive constant, the Bochner Laplacian defined on the complex line bundles over S^2 (Kuwabara 1982). Hence it turns out that the spherical part of the eigen cross section for the quantised MIC-Kepler Hamiltonian $\hat{H}_m$ gives the monopole harmonics. Local expressions of the eigen cross section for $\hat{H}_m$ are also given in the spherical coordinates in $\dot{\boldsymbol{R}}^3$.

2. A review of the quantised conformal Kepler problem

We review in this section the quantised conformal Kepler problem defined in Iwai (1982a). The quantised conformal Kepler problem is defined as a pair $(L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x), \hat{H})$. Here, $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ is the Hilbert space of square integrable complex-valued functions on $\boldsymbol{R}^4$ with respect to the inner product

$$\langle f, g\rangle = \int_{\boldsymbol{R}^4} \bar{f} g 4r\,\mathrm{d}x \tag{3}$$

where $\bar{f}$ is the complex conjugate of f, $r=\Sigma_{j=1}^{j=4} x_j^2$, and $\mathrm{d}x=\mathrm{d}x_1\ldots\mathrm{d}x_4$. The $\hat{H}$ is the Hamiltonian operator given by

$$\hat{H} = -\frac{1}{2}\left(\frac{1}{4r}\sum_{j=1}^{4}\frac{\partial^2}{\partial x_j^2}\right) - \frac{k}{r}. \tag{4}$$

$\hat{H}$ is a symmetric operator on $C_0^\infty(R^4)$, the space of smooth functions of compact support, with respect to the inner product (3). Note also that $C_0^\infty(\boldsymbol{R}^4)$ is dense in $L^2(R^4; 4r\,\mathrm{d}x)$.

The operator $\hat{H}$ is related to the harmonic oscillator Hamiltonian operator $\hat{K}$

$$\hat{K} = -\frac{1}{2}\sum_{j=1}^{4}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}\lambda^2 r \qquad (\lambda > 0, \text{ constant}). \tag{5}$$

In fact, one has, from (4) and (5),

$$4r(\hat{H} + \lambda^2/8) = \hat{K} - 4k. \tag{6}$$

This relation makes it possible to approach the quantised conformal Kepler problem through the harmonic oscillator. Indeed, eigenfunctions of $\hat{H}$ of negative energy can be looked upon as eigenfunctions of $\hat{K}$. The work of Kibler and Negati (1983, 1984a, b) is in an analogous line of thought.

For later use, we here introduce harmonic oscillator annihilation and creation operators a_j and a_j^+, respectively, by

$$\begin{aligned} a_j &= (2\lambda)^{-1/2}(\lambda x_j + \partial/\partial x_j) \\ a_j^+ &= (2\lambda)^{-1/2}(\lambda x_j - \partial/\partial x_j). \end{aligned} \tag{7}$$

The commutation relations among them are

$$[a_j, a_k^+] = \delta_{jk} \qquad \text{the others vanish.} \tag{8}$$

The normalised eigenfunctions for the harmonic oscillator are then expressed in the form

$$\psi_{\boldsymbol{l}}(x) = (\boldsymbol{l}!)^{-1/2}(a_1^+)^{l_1}(a_2^+)^{l_2}(a_3^+)^{l_3}(a_4^+)^{l_4}\psi_0(x) \tag{9a}$$

$$\boldsymbol{l}! = l_1!\,l_2!\,l_3!\,l_4! \tag{9b}$$

where the bold subscript $\boldsymbol{l}$ is a multi-index denoting $(l_1 l_2 l_3 l_4)$ and $l_1, \ldots, l_4$ are non-negative integers. The function $\psi_0(x)$ is the normalised ground state $\psi_0(x) = (\lambda/\pi)^{1/2}\exp(-\lambda r/2)$. The eigenspace associated with the eigenvalue $\lambda(n+2)$ is spanned by those functions with $l_1 + l_2 + l_3 + l_4 = n$.

This eigenspace can be put into an eigenspace of the quantised conformal Kepler problem. Setting $4k = \lambda(n+2)$, we see from (6) that the corresponding eigenvalues of $\hat{H}$ become $-\lambda^2/8 = -2k^2/(n+2)^2$. Thus we have the following (Iwai 1982a).

Proposition 2.1. The negative eigenvalues of the quantised conformal Kepler problem are $E_n := -2k^2/(n+2)^2$, n being non-negative integers, and the corresponding eigenfunctions are given by (9) with $\lambda = 4k/(n+2)$ and $l_1 + l_2 + l_3 + l_4 = n$.

3. Reduction of $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$

In this section we reduce $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ by an S^1 action (the definition of $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ was given in § 2). This is a quantum version of the reduction we performed in Iwai and Uwano (1986) for the classical conformal Kepler problem. We recall here that the point in the reduction of the conformal Kepler problem is to observe that the configuration space $\dot{\boldsymbol{R}}^4 = \boldsymbol{R}^4 - \{0\}$ is a principal U(1) bundle; $\pi: \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3 = \boldsymbol{R}^3 - \{0\}$. This suggests that the reduced quantum system must be defined in a Hilbert space of 'wavefunctions' on $\dot{\boldsymbol{R}}^3$. To be precise, wavefunctions should be generalised to 'wavesections' in a complex line bundle over $\dot{\boldsymbol{R}}^3$.

We start by reviewing the principal U(1) bundle $\pi: \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$. Let Φ_t be an S^1 action on $\dot{\boldsymbol{R}}^4$ defined by a 4×4 matrix $T(t)$:

$$\Phi_t x = T(t)x \qquad x \in \dot{\boldsymbol{R}}^4 \qquad t \in [0, 4\pi] \tag{10a}$$

and

$$T(t) = \begin{pmatrix} R(t) & \\ & R(t) \end{pmatrix} \qquad R(t) = \begin{pmatrix} \cos t/2 & -\sin t/2 \\ \sin t/2 & \cos t/2 \end{pmatrix}. \tag{10b}$$

Here and henceforce missing matrix entries are all zero. Then the U(1) bundle is defined as a natural projection $\pi: \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^4/S^1 \simeq \dot{\boldsymbol{R}}^3$. In an explicit manner, the π is given by

$$\begin{aligned} \tilde{x}_1 &= 2(x_1x_3 + x_2x_4) \\ \tilde{x}_2 &= 2(-x_1x_4 + x_2x_3) \\ \tilde{x}_3 &= x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{aligned} \tag{11}$$

where $(\tilde{x}_j)$ are the cartesian coordinates in $\dot{\boldsymbol{R}}^3$. We note here that if $\dot{\boldsymbol{R}}^4$ is restricted onto the unit sphere S^3 the mapping π becomes the Hopf fibring $S^3 \to S^2$ (cf Ryder 1980), which is easily seen from

$$\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 = \left(\sum_{j=1}^{4} x_j^2\right)^2.$$

The Φ_t can be lifted to act symplectically on the cotangent bundle $T^*\dot{\boldsymbol{R}}^4$. By using the momentum mapping associated with the lifted S^1 action, the reduction of the conformal Kepler problem was carried out (Iwai and Uwano 1986). We have here to consider a quantum analogue to that reduction. In quantum mechanics, the momentum operator associated with Φ_t is defined by

$$\hat{N}f = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t} f(T(-t)x)\bigg|_{t=0}. \tag{12a}$$

A calculation gives

$$\hat{N} = \tfrac{1}{2}\mathrm{i}(-x_2\partial/\partial x_1 + x_1\partial/\partial x_2 - x_4\partial/\partial x_3 + x_3\partial/\partial x_4). \tag{12b}$$

We denote the self-adjoint extension of $\hat{N}$ on a certain domain in $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ by the same letter.

To fix a momentum eigenvalue of $\hat{N}$ then amounts to a restriction of the Hilbert space $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ to an eigenspace of $\hat{N}$. This procedure corresponds to fixing the momentum value in the classical case. However, instead of considering the operator $\hat{N}$, we treat the unitary operator U_t on $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ defined through

$$(U_t f)(x) = f(T(-t)x). \tag{13}$$

We notice here that the infinitesimal generator of U_t is, of course, $\mathrm{i}\hat{N}$, and that the volume element $4r\,\mathrm{d}x$ is invariant under Φ_t. Then, by using the U_t, we can say what subspace of $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ is replaced for the eigenspace of $\hat{N}$.

To start with, we introduce complex line bundles. Let ρ_m denote a unitary irreducible representation of U(1) (or Φ_t) on $\boldsymbol{C}$ defined by

$$\rho_m(\Phi_t)\colon z \to \exp(\mathrm{i}mt/2)z \tag{14}$$

where m is an arbitrary integer and t ranges from 0 to 4π. Let U(1) act on $\dot{\boldsymbol{R}}^4 \times \boldsymbol{C}$ to the left in the form

$$(x, z) \to (T(t)x, \exp(\mathrm{i}mt/2)z) \qquad x \in \dot{\boldsymbol{R}}^4 \qquad z \in \boldsymbol{C}. \tag{15}$$

Then the quotient manifold denoted by $\dot{\boldsymbol{R}}^4 \times_m \boldsymbol{C}$ is made into a complex line bundle $L_m = (\dot{\boldsymbol{R}}^4 \times_m \boldsymbol{C}, \pi_m, \dot{\boldsymbol{R}}^3)$, where π_m is the projection, $\pi_m : \dot{\boldsymbol{R}}^4 \times_m \boldsymbol{C} \to \dot{\boldsymbol{R}}^3$. The L_m is called the complex line bundle associated with the U(1) bundle $\pi : \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$. Denoting by q the natural projection $\dot{\boldsymbol{R}}^4 \times \boldsymbol{C} \to \dot{\boldsymbol{R}}^4 \times_m \boldsymbol{C}$, one has the commutative diagram

$$\begin{tikzcd} \dot{\boldsymbol{R}}^4 \times \boldsymbol{C} \arrow[r, "q"] \arrow[d, "p_1"] & \dot{\boldsymbol{R}}^4 \times_m \boldsymbol{C} \arrow[d, "\pi_m"] \\ \dot{\boldsymbol{R}}^4 \arrow[r, "\pi"] & \dot{\boldsymbol{R}}^3 \end{tikzcd} \tag{16}$$

where p_1 is the projection onto the first factor. We note here that we are treating the left action of U(1), contrary to the usual definition of the associated bundle (Greub and Petry 1975).

A complex-valued function f is called ρ_m equivariant if it satisfies

$$f(T(t)x) = \exp(\mathrm{i}mt/2)f(x). \tag{17}$$

It is well known that ρ_m-equivariant functions on $\dot{\boldsymbol{R}}^4$ are in one-to-one correspondence with cross sections in L_m (Greub and Petry 1975, Kobayashi and Nomizu 1963). We denote by q_m the correspondence; for a ρ_m-equivariant function f and the corresponding cross section γ in L_m we have $q_m f = \gamma$. Further, for a (smooth) ρ_m-equivariant function f, we obtain, by differentiating (17) with respect to t,

$$\hat{N}f = -(m/2)f. \tag{18}$$

Thus f turns out to be an eigenfunction of $\hat{N}$ which corresponds uniquely to a cross section in L_m. Accordingly, the introduction of L_m is understood as a geometric consequence of the conservation of the angular momentum associated with the U(1) action (10).

Let the γ be cross sections in L_m such that $q_m^{-1}\gamma$ are in $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$. Then $q_m^{-1}\gamma$ are ρ_m equivariant, so that the inner product in $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ gives rise to the inner product in the space of such cross sections:

$$\langle \gamma_1, \gamma_2 \rangle_m = \frac{1}{4\pi} \int_{\boldsymbol{R}^4} \overline{(q_m^{-1}\gamma_1)(x)}(q_m^{-1}\gamma_2)(x) 4r\,\mathrm{d}x \tag{19}$$

where the bar indicates the complex conjugate. We note here that the integral is invariant under the U(1) action. Thus the space of the ρ_m-equivariant functions in $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ is associated with the Hilbert space, denoted by Γ_m, of square integrable cross sections in L_m. In other words, $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ is reduced to Γ_m by restricting $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ to the functions satisfying $U_t f = \exp(-\mathrm{i}mt/2)f$. This finishes the reduction.

We recall here that the classical reduced system of the conformal Kepler problem has Dirac's monopole field (Iwai and Uwano 1986), so that the reduced quantum system to be defined on Γ_m must carry Dirac's monopole field. This requirement, however, can be fulfilled by introducing a linear connection on the complex line bundle L_m.

In what follows we study the linear connection on L_m which is induced from a connection on the principal fibre bundle $\pi : \dot{\boldsymbol{R}}^4 \to \dot{\boldsymbol{R}}^3$. The S^1 action Φ_t generates a vector field N on $\dot{\boldsymbol{R}}^4$; $N=-\mathrm{i}\hat{N}$. At every point x of $\dot{\boldsymbol{R}}^4$, a horizontal subspace V_x of $T_x(\dot{\boldsymbol{R}}^4)$ is defined as the set of all tangent vectors orthogonal to N_x with respect to the standard inner product on $\dot{\boldsymbol{R}}^4$ or with respect to the conformally flat metric $\mathrm{d}s_c^2$. It is easy to verify that the assignment of V_x to every $x\in\dot{\boldsymbol{R}}^4$ is a connection on $\dot{\boldsymbol{R}}^4$ (Greub and Petry 1975). To every vector field X on $\dot{\boldsymbol{R}}^3$ there is a unique horizontal vector field X^* on $\dot{\boldsymbol{R}}^4$ satisfying

$$\pi_* X_x^* = X_{\pi(x)} \qquad x\in\dot{\boldsymbol{R}}^4 \tag{20}$$

where π_* is the tangent map of π. The X^* is called the horizontal lift of X.

Using q_m, we can define a linear connection ∇ on L_m by setting (Greub and Petry 1975)

$$\nabla_X\gamma = q_m(X^* q_m^{-1}\gamma) \tag{21}$$

for a cross section γ in L_m and a vector field X on $\dot{\boldsymbol{R}}^3$. By definition, the curvature R of the connection ∇ is given by

$$R(X,Y)\gamma = ([\nabla_X,\nabla_Y]-\nabla_{[X,Y]})\gamma. \tag{22}$$

Combined with (21), the definition (22) is put into the form

$$R(X,Y)\gamma = q_m[([X^*,Y^*]-[X,Y]^*)q_m^{-1}\gamma]. \tag{23}$$

Using (23), we can write out the curvature R. We note that the horizontal lifts of $\partial/\partial\tilde{x}_j$, $j=1,2,3$, are given by

$$\begin{pmatrix}(\partial/\partial\tilde{x}_1)^*\\(\partial/\partial\tilde{x}_2)^*\\(\partial/\partial\tilde{x}_3)^*\end{pmatrix}=\frac{1}{2r}\begin{pmatrix}x_3 & x_4 & x_1 & x_2\\-x_4 & x_3 & x_2 & -x_1\\x_1 & x_2 & -x_3 & -x_4\end{pmatrix}\begin{pmatrix}\partial/\partial x_1\\\partial/\partial x_2\\\partial/\partial x_3\\\partial/\partial x_4\end{pmatrix}. \tag{24}$$

Calculation then yields

$$[(\partial/\partial\tilde{x}_1)^*,(\partial/\partial\tilde{x}_2)^*]=(\tilde{x}_3/r^3)N \qquad \text{(cyclic)}. \tag{25}$$

From (23) and (25) it follows that

$$R(\partial/\partial\tilde{x}_1,\partial/\partial\tilde{x}_2)=(\mathrm{i}m/2)(\tilde{x}_3/r^3) \qquad \text{(cyclic)} \tag{26}$$

where we have used the fact that $q_m^{-1}\gamma$ is a ρ_m-equivariant function. Equation (26) is written in the form

$$R(X,Y)\gamma = -\mathrm{i}(-m/2)\Omega(X,Y)\gamma \tag{27}$$

where X and Y are vector fields on $\dot{\boldsymbol{R}}^3$, and

$$\Omega = r^{-3}(\tilde{x}_1\,\mathrm{d}\tilde{x}_2\wedge\mathrm{d}\tilde{x}_3+\tilde{x}_2\,\mathrm{d}\tilde{x}_3\wedge\mathrm{d}\tilde{x}_1+\tilde{x}_3\,\mathrm{d}\tilde{x}_1\wedge\mathrm{d}\tilde{x}_2) \qquad r^2=\sum\tilde{x}_j^2. \tag{28}$$

This shows that the curvature R defines Dirac's monopole field of strength $-m/2$. Thus we have taken Dirac's monopole field into L_m.

The above discussion is summed up in the following theorem.

Theorem 3.1. By an S^1 action, the Hilbert space $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$ is reduced to the Hilbert space Γ_m, m being an integer, of square integrable cross sections in the complex line bundles L_m over $\dot{\boldsymbol{R}}^3$. The L_m is endowed with the linear connection whose curvature gives Dirac's monopole field of strength $-m/2$. If $m=0$, the L_m becomes a trivial bundle $\dot{\boldsymbol{R}}^3\times\boldsymbol{C}$, and Dirac's monopole field vanishes.

In quantum mechanics, magnetic fields should appear through vector potentials. To check this, we investigate the connection ∇ in terms of the coordinates $(\tilde{x}_j)$. Consider a locally defined ρ_m-equivariant function

$$\chi_+(x) = [(x_1 + \mathrm{i}x_2)(x_1^2 + x_2^2)^{-1/2}]^m. \tag{29}$$

This defines a local cross section $q_m\chi_+ = \gamma_+$ in L_m which is defined in

$$U_+ = \{\tilde{x} \in \dot{\boldsymbol{R}}^3 \text{ without negative } \tilde{x}_3 \text{ axis}\}. \tag{30}$$

Then any local cross section defined on U_+ can be expressed as $f^+\gamma_+$, where f^+ is a complex-valued function on U_+.

For the local cross section γ_+, we can calculate the covariant derivative $\nabla_X\gamma_+$, using (21). Denoting by ∇_j the covariant derivation $\nabla_{\partial/\partial\tilde{x}_j}$, we obtain, after calculation,

$$\begin{aligned}
&\nabla_1\gamma_+ = \mathrm{i}\frac{m}{2}\frac{-\tilde{x}_2}{r(r+\tilde{x}_3)}\gamma_+ \\
&\nabla_2\gamma_+ = \mathrm{i}\frac{m}{2}\frac{\tilde{x}_1}{r(r+\tilde{x}_3)}\gamma_+ \\
&\nabla_3\gamma_+ = 0.
\end{aligned} \tag{31}$$

Let A^+ be a vector field on U_+ defined by

$$A^+ = \left(\frac{-\tilde{x}_2}{r(r+\tilde{x}_3)}, \frac{\tilde{x}_1}{r(r+\tilde{x}_3)}, 0\right) \tag{32}$$

which is a locally defined vector potential used in McIntosh and Cisneros (1970). Then (31) is expressed as

$$\nabla_j\gamma_+ = -\mathrm{i}(-m/2)A_j^+\gamma_+ \qquad j = 1, 2, 3. \tag{33}$$

The covariant derivative of $f^+\gamma_+$ then turns out to be

$$\nabla_j(f^+\gamma_+) = [\partial f^+/\partial\tilde{x}_j - \mathrm{i}(-m/2)A_j^+f^+]\gamma_+ \tag{34}$$

which is often used in the literature in treating Dirac's monopole field in quantum mechanics.

If we take another local ρ_m-equivariant function

$$\chi_-(x) = [(x_3 + \mathrm{i}x_4)(x_3^2 + x_4^2)^{-1/2}]^m \tag{35}$$

we will obtain a local expression of ∇_j different from (34) but we do not go into detail. Another local expression in curvilinear coordinates will be given in § 8.

4. Reduction of the quantised conformal Kepler problem

In this section we wish to study what Hamiltonian operator should be derived in the reduced Hilbert space Γ_m from the Hamiltonian operator (4). For this purpose, we start with a one-parameter group of unitary operators U_t defined by (13). The operators (x_j) and $(\partial/\partial x_j)$ are then subject to the transformations

$$U_t x U_t^{-1} = T(-t)x \tag{36}$$

$$U_t(\partial/\partial x)U_t^{-1} = T(-t)\partial/\partial x \tag{37}$$

respectively, where x and $\partial/\partial x$ are column vectors. We here restrict domains of x_j and $\partial/\partial x_j$, $\hat{H}$ and so on, say, to $C_0^\infty(\boldsymbol{R}^4)$, the space of smooth functions of compact support.

Using (36) and (37), we can prove that the Hamiltonian operator $\hat{H}$ is invariant under U_t; $U_t\hat{H}U_t^{-1}=\hat{H}$. In fact, $\hat{H}$ is composed of $\Sigma\,\partial^2/\partial x_j^2$ and $\Sigma\, x_j^2$ which are invariant under U_t.

To reduce $\hat{H}$, we have only to restrict the domain of $\hat{H}$ to ρ_m-equivariant functions. For a ρ_m-equivariant function f, we can define a reduced operator $\hat{H}_m$ by

$$\hat{H}_m(q_m f)=q_m(\hat{H}f). \tag{38}$$

In fact, $\hat{H}f$ is ρ_m equivariant, i.e. $U_t\hat{H}f=\exp(-\mathrm{i}mt/2)\hat{H}f$, because of $U_t\hat{H}U_t^{-1}=\hat{H}$ and $U_tf=\exp(-\mathrm{i}mt/2)f$.

In what follows we describe $\hat{H}_m$ in an explicit manner. In terms of the horizontal lifts $(\partial/\partial\tilde{x}_j)^*$ given in (24) and the momentum operator $\hat{N}$ given in (12b), the Hamiltonian operator $\hat{H}$ is expressed as

$$\hat{H}=-\frac{1}{2}\sum_{j=1}^{3}\left[\left(\frac{\partial}{\partial\tilde{x}_j}\right)^*\right]^2+(1/2r^2)\hat{N}^2-(k/r). \tag{39}$$

An analogous relation to this is found in Kibler and Negadi (1983). We operate a ρ_m-equivariant function f with $\hat{H}$, and $\hat{H}f$ with q_m, to get

$$\hat{H}_m\gamma=\left(-\frac{1}{2}\sum_{j=1}^{3}\nabla_j^2+\frac{(m/2)^2}{2r^2}-\frac{k}{r}\right)\gamma \tag{40}$$

where $\gamma=q_mf$. We have here used (18) and (21). A comparison of (40) with (1) shows that the $\hat{H}_m$ is a quantisation of H_μ by replacing $\tilde{p}_j$ and μ with $-\mathrm{i}\nabla_j$ and $-m/2$, respectively. Thus we obtain the following theorem.

Theorem 4.1. The quantised conformal Kepler problem $(L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x), \hat{H})$ is reduced to the quantum system $(\Gamma_m, \hat{H}_m)$, where Γ_m is the Hilbert space in theorem 3.1, and $\hat{H}_m$ is given by (40). We refer to $(\Gamma_m, \hat{H}_m)$ as the quantised MIC-Kepler problem. If $m=0$, the reduced system becomes the hydrogen atom $(L^2(\boldsymbol{R}^3), \hat{H}_0)$ (Iwai 1982a).

In conclusion, we give local expression to $\hat{H}_m$. Let $f^+\gamma_+$ be a local cross section as in § 3. Then (40) turns out to be

$$\hat{H}_m(f^+\gamma_+)=\left[-\frac{1}{2}\sum_{j=1}^{3}\left(\frac{\partial}{\partial\tilde{x}_j}-\mathrm{i}(-m/2)A_j^+\right)^2+\frac{(m/2)^2}{2r^2}-\frac{k}{r}\right]f^+\gamma_+. \tag{41}$$

This shows that the $\hat{H}_m$ is indeed the Hamiltonian for a particle moving in the presence of Dirac's monopole field of strength $-m/2$, a centrifugal potential $(m/2)^2/2r^2$ and a Newtonian potential $-k/r$.

5. Negative-energy eigenspaces of $(\Gamma_m, \hat{H}_m)$

The negative-energy eigenfunctions for $(\Gamma_m, \hat{H}_m)$ can be obtained from those for $(L^2(\boldsymbol{R}^4, 4r\,\mathrm{d}x), \hat{H})$ by the reduction. To carry out this is the purpose of this section. It is convenient to introduce the creation and annihilation operators b_j^+ and b_j, $j=1,2,3,4$, by

$$\begin{aligned} b_1^+&=(a_1^+-\mathrm{i}a_2^+)/\sqrt{2} \qquad & b_2^+&=(a_3^+-\mathrm{i}a_4^+)/\sqrt{2}\\ b_3^+&=(a_1^++\mathrm{i}a_2^+)/\sqrt{2} \qquad & b_4^+&=(a_3^++\mathrm{i}a_4^+)/\sqrt{2} \end{aligned} \tag{42}$$

together with the adjoint relations. It is easy to check that

$$[b_j, b_k^+] = \delta_{jk} \qquad \text{the others vanish.} \tag{43}$$

Then, in terms of b_j^+, one can define another set of the harmonic oscillator eigenfunctions (cf (9))

$$\begin{gathered}\Psi_{\boldsymbol{k}}(x) = (\boldsymbol{k}!)^{-1/2}(b_1^+)^{k_1}(b_2^+)^{k_2}(b_3^+)^{k_3}(b_4^+)^{k_4}\psi_0(x) \\ \boldsymbol{k}! := k_1!\,k_2!\,k_3!\,k_4!.\end{gathered} \tag{44}$$

As in (9), the bold subscript $\boldsymbol{k}$ is a multi-index denoting $(k_1k_2k_3k_4)$, where $k_1, \ldots, k_4$ are non-negative integers.

The eigenspace, denoted by $S(E_n)$, of negative energy $E_n = -2k^2/(n+2)^2$ for the conformal Kepler problem is then spanned by (44) with $\lambda = 4k/(n+2)$ and $k_1 + k_2 + k_3 + k_4 = n$ (see proposition 2.1).

The introduction of the operators b_j and b_j^+ makes it easy to pick out the ρ_m-equivariant functions from these eigenfunctions. We note indeed that the unitary operator U_t induces the transformations of b_j^+ in the form

$$U_t b^+ U_t^{-1} = T'(-t)b^+ \qquad T'(-t) := \begin{pmatrix} e^{it/2}I_2 & \\ & e^{-it/2}I_2 \end{pmatrix} \tag{45}$$

where b^+ is a column vector of (b_j^+) and the missing matrix entries are all zero, I_2 being the 2×2 identity matrix. From (45) and the fact that $\psi_0(x)$ is U_t invariant, the eigenfunctions (44) are subject to the transformation

$$(U_t\Psi_{\boldsymbol{k}})(x) = \exp[i(k_1 + k_2 - k_3 - k_4)t/2]\Psi_{\boldsymbol{k}}(x). \tag{46}$$

Thus the U_t is diagonalised with respect to (44). Therefore, it turns out that eigenfunctions $\Psi_{\boldsymbol{k}}(x)$ of energy E_n are ρ_m equivariant if and only if the multi-index $\boldsymbol{k}$ satisfies

$$k_1 + k_2 + k_3 + k_4 = n \tag{47a}$$

$$-k_1 - k_2 + k_3 + k_4 = m. \tag{47b}$$

Thus one has

$$k_1 + k_2 = (n - m)/2 \qquad k_3 + k_4 = (n + m)/2. \tag{48}$$

Since k_j are non-negative integers, m and n must satisfy

$$|m| \leqslant n \tag{49a}$$

and, moreover,

$$m \text{ and } n \text{ are simultaneously even or odd.} \tag{49b}$$

These conditions were pointed out in McIntosh and Cisneros (1970) by solving the Schrödinger equation $\hat{H}_m\gamma = E\gamma$ in local coordinates. By $S(E_n; m)$ we denote the subspace of $S(E_n)$ consisting of ρ_m-equivariant eigenfunctions of energy E_n. Then the dimension of $S(E_n; m)$ is determined by (48):

$$\dim S(E_n; m) = (n - m + 2)(n + m + 2)/4. \tag{50}$$

We are now in a position to describe eigenspaces of negative energy for the quantised MIC-Kepler problem $(\Gamma_m, \hat{H}_m)$. Let f be in $S(E_n; m)$. Then $\hat{H}f = E_nf$. Applying (38) to this, we have

$$\hat{H}_m(q_mf) = E_n(q_m f). \tag{51}$$

This means that the cross section $q_m f$ in $\mathrm{L_m}$ is an eigen cross section of $\hat{H}_m$. Conversely, if γ is an eigen cross section of $\hat{H}_m$, the ρ_m-equivariant function $q_m^{-1}\gamma$ is an eigenfunction of $\hat{H}$. Therefore, all the negative eigenvalues of $\hat{H}_m$ are given by E_n, subject to (49), and the eigenspaces are given by $q_m S(E_n\,;\,m)$. In summary, we have the following.

Theorem 5.1. The ρ_m-equivariant eigensubspace $S(E_n\,;\,m)$ for the quantised conformal Kepler problem is in one-to-one correspondence with the eigenspace $q_m S(E_n\,;\,m)$ of negative energy $E_n=-2k^2/(n+2)^2$ for the quantised MIC-Kepler problem $(\Gamma_m, \hat{H}_m)$, where n and m are subject to the conditions (49). The $q_m S(E_n\,;\,m)$ is of dimension $(n-m+2)(n+m+2)/4$.

6. The symmetry group for $(\Gamma_m, \hat{H}_m)$ of negative energy

This section shows that the eigenspace $q_m S(E_n\,;\,m)$ of the quantised MIC-Kepler problem $(\Gamma_m, \hat{H}_m)$ admits a group action which accounts for the degeneracy of the energy level E_n.

According to Iwai (1982b), SU(4) acts on $L^2(\boldsymbol{R}^4)$ unitarily as follows. Let U_C be a unitary operator corresponding to a matrix C in SU(4). The action of U_C on the complete basis Ψ_k of $L^2(\boldsymbol{R}^4)$ is then given by

$$U_C\Psi_k(x)=(k!)^{-1/2}(C^{\mathrm{T}}b^+)_1^{k_1}\ldots(C^{\mathrm{T}}b^+)_4^{k_4}\psi_0(x) \tag{52}$$

where b^+ is a column vector of b_j^+ and $(C^{\mathrm{T}}b^+)_j$ denotes the jth component of the vector $C^{\mathrm{T}}b^+$, the superscript T meaning the transpose. Note that the U_C induces the transformation of b^+

$$U_C b^+ U_C^{-1}=C^{\mathrm{T}}b^+. \tag{53}$$

We mention here that the operator (52) covers the unitary operator U_t defined by (13), if C is replaced by $T'(-t)$ given in (45).

Restricted to the eigenspace for the harmonic oscillator ($k_1+k_2+k_3+k_4=n$), (52) gives a unitary irreducible representation of SU(4), because each eigenspace is identified with the vector space of homogeneous polynomials of degree n. If we set $\lambda=4k/(n+2)$ in (7) and therefore in (42), this representation gives rise to an action of SU(4) on the negative-energy eigenspace $S(E_n)$ of the quantised conformal Kepler problem of eigenvalue E_n. We denote this action by $U_C^{(n)}$. By carrying out a straightforward calculation the $U_C^{(n)}$ can be shown to be unitary in $S(E_n)$ with respect to the inner product (3). The proof of this unitarity is presented in the appendix.

We now wish to know what subgroup of SU(4) acts on $S(E_n\,;\,m)$, the subspace of $S(E_n)$ consisting of ρ_m-equivariant eigenfunctions of negative eigenvalue E_n. To this end, we show the following lemma.

Lemma 6.1. $S(E_n\,;\,m)$ is an invariant subspace for $U_C^{(n)}$ if and only if $U_C^{(n)}$ and U_t commute.

Proof. By $\Psi_k^{(n,m)}$, we denote $\Psi_k(x)$ that is in $S(E_n\,;\,m)$, from now on, when we want to emphasise that Ψ_k is in $S(E_n\,;\,m)$. If $U_C^{(n)}$ acts on $S(E_n\,;\,m)$, $U_C^{(n)}\Psi_k^{(n,m)}$ must be

ρ_m equivariant for any $\Psi_k^{(n,m)}$. Thus we have

$$\begin{aligned} U_t(U_C^{(n)}\Psi_k^{(n,m)}) &= \exp(-\mathrm{i}mt/2)(U_C^{(n)}\Psi_k^{(n,m)}) \\ &= U_C^{(n)}[\exp(-\mathrm{i}mt/2)\Psi_k^{(n,m)}] \\ &= U_C^{(n)}(U_t\Psi_k^{(n,m)}). \end{aligned} \tag{54}$$

The converse can be proved in an analogous manner. This ends the proof.

Suppose $U_C^{(n)}$ acts on $S(E_n\,;\,m)$, and let $\gamma_k^{(n,m)} := q_m\Psi_k^{(n,m)}$, for $\Psi_k^{(n,m)} \in S(E_n\,;\,m)$. Then, by virtue of theorem 5.1, we can define an action $U_C^{(n,m)}$, from $U_C^{(n)}$, on the eigenspace $q_mS(E_n\,;\,m)$ of the quantised MIC-Kepler problem by

$$U_C^{(n,m)}\gamma_k^{(n,m)} := q_mU_C^{(n)}q_m^{-1}\gamma_k^{(n,m)}. \tag{55}$$

Thus a subgroup of SU(4) acting on $S(E_n\,;\,m)$ becomes a symmetry group acting on $q_mS(E_n\,;\,m)$, the eigenspace for $\hat{H}_m$.

Consequently, from lemma 6.1, in order to obtain the symmetry group, we have only to find matrices C which commute with $T'(t)$. We notice again that when restricted on $S(E_n\,;\,m)$ the U_t is expressed in the form $U_{T'(-t)}^{(n)}$. Now a straightforward calculation shows that the matrices commuting with $T'(t)$ take the form

$$C = \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix} \tag{56}$$

where $C_1, C_2 \in \mathrm{U}(2)$ with $\det C_1 \det C_2 = 1$, and the missing matrix entries are all zero. Hence, the subgroup commuting with $T'(t)$ becomes $S(\mathrm{U}(2)\times\mathrm{U}(2))$, where S means 'unimodular'. This result is the same as in the classical system (Iwai and Uwano 1986). We wish to treat $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$, a double covering group of $S(\mathrm{U}(2)\times\mathrm{U}(2))$, rather than $S(\mathrm{U}(2)\times\mathrm{U}(2))$ itself. The projection of $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ to $S(\mathrm{U}(2)\times \mathrm{U}(2))$ is given by

$$C = T'(t)\begin{pmatrix} C_1' & \\ & C_2' \end{pmatrix} = \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix} \tag{57}$$

where $T'(t) \in \mathrm{U}(1)$ and $C_1', C_2' \in \mathrm{SU}(2)$. Since U(1) is represented as a scalar multiplication by $\exp(-\mathrm{i}mt/2)$ on $S(E_n\,;\,m)$, we consider instead the action of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ on $S(E_n\,;\,m)$ in detail.

Then from (52) and (57) we see that the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ action is isomorphic with a tensor product of irreducible representations of SU(2). To stress this point of view, it is helpful to denote b_1^+, b_2^+, b_3^+, and b_4^+ by A_1^+, A_2^+, B_1^+ and B_2^+, respectively. Then, the basis functions $\Psi_k^{(n,m)}$ in $S(E_n\,;\,m)$ are expressed as

$$\Psi_k^{(n,m)} = (\boldsymbol{k}!)^{-1/2}(A_1^+)^{k_1}(A_2^+)^{k_2}(B_1^+)^{k_3}(B_2^+)^{k_4}\psi_0(x) \tag{58}$$

where the multi-index $\boldsymbol{k}$ is subject to (48). From the condition (48), one sees that $S(E_n\,;\,m)$ can be identified with the tensor product of the space of homogeneous polynomials in (A_j^+) of degree $(n-m)/2$ and the one of homogeneous polynomials in (B_j^+) of degree $(n+m)/2$. Then the $S(E_n\,;\,m)$ carries a unitary representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$; from (52), (57) and (58), one has

$$(\boldsymbol{k}!)^{-1/2}(C_1^{\mathrm{T}}A^+)_1^{k_1}(C_1^{\mathrm{T}}A^+)_2^{k_2}(C_2^{\mathrm{T}}B^+)_1^{k_3}(C_2^{\mathrm{T}}B^+)_2^{k_4}\psi_0(x) \tag{59a}$$

with

$$C = \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix} \in \mathrm{SU}(2)\times\mathrm{SU}(2) \qquad C_1, C_2 \in \mathrm{SU}(2). \tag{59b}$$

Here, A^+ and B^+ are column vectors of A_j^+ and $B_j^+, j=1,2$, respectively. $(C_1^{\mathrm{T}}A^+)_j$ and $(C_2^{\mathrm{T}}B^+)_j$ denote the jth component of the vectors $C_1^{\mathrm{T}}A^+$ and $C_2^{\mathrm{T}}B^+$, respectively, the superscript T meaning the transpose. Since the factor groups SU(2) of SU(2) × SU(2) are both represented irreducibly in $(n-m)/2$-degree (respectively $(n+m)/2$-degree) homogeneous polynomial space of A^+ (respectively B^+), the representation (59) becomes a unitary irreducible representation of SU(2) × SU(2). It should be noted that the representation (59) exhausts all the unitary irreducible representations (cf Wigner 1959). These representations give rise to unitary actions of SU(2) × SU(2), $U_C^{(n,m)}$, on $q_m S(E_n; m)$ through (55) together with (59*b*). Thus we have realised all the unitary irreducible representations, up to the equivalence, in $q_m S(E_n; m)$, every eigenspace for the Hamiltonian operator $\hat{H}_m$ acting on the complex line bundle L_m.

We proceed further to discuss the symmetry group SU(2) × SU(2). Recalling that the integers n and m are subject to the condition (49*b*), we break up our discussion into two: n and m are simultaneously (i) even and (ii) odd. We now note that for $(-I_2, -I_2)$ of SU(2) × SU(2) one has

$$U^{(n,m)}_{(-I_2,-I_2)}\gamma_k^{(n,m)} = (-1)^n \gamma_k^{(n,m)}. \tag{60}$$

This implies that, according to whether n is even or odd, the right-hand side of (60) is equal to $\gamma_k^{(n,m)}$ or to $-\gamma_k^{(n,m)}$. Thus in case of (i) we should understand that $\mathrm{SU}(2)\times\mathrm{SU}(2)/\boldsymbol{Z}_2 \simeq \mathrm{SO}(4)$ is represented irreducibly in $q_m S(E_n; m)$, where $\boldsymbol{Z}_2$ consists of (I_2, I_2) and $(-I_2, -I_2)$. In the case of (ii), SU(2) × SU(2) is represented irreducibly, of course. We turn our attention to the condition (49*a*), especially to the case $|m| = n$. Then, (48) gives, if m is positive,

$$k_1 + k_2 = 0 \qquad k_3 + k_4 = m. \tag{61}$$

Therefore, one has $k_1 = k_2 = 0$. From this and (59*a*) it follows that the action of the first factor of SU(2) × SU(2) becomes trivial, so that the action of the second factor is significant. The same discussion can run through, if m is negative, to result in showing the first factor is significant. Further, if m is even, (60) also shows that $-I_2$ of SU(2) fixes all the elements of (61), so that $\mathrm{SU}(2)/\boldsymbol{Z}_2 \simeq \mathrm{SO}(3)$ is considered as represented unitarily irreducibly on $S(E_{|m|}; m)$, where $\boldsymbol{Z}_2 = \{I_2, -I_2\}$. If m is odd, SU(2) is represented unitarily irreducibly.

Summing up the above, we have the following.

Theorem 6.2. Under conditions (49), the negative-energy eigenspace $q_m S(E_n; m)$ of the quantised MIC-Kepler problem $(\Gamma_m, \hat{H}_m)$ admits the unitary irreducible representations of the symmetry group

(i) SO(4) if m is even

(ii) SU(2) × SU(2) if m is odd.

All the unitary irreducible representations of SU(2) × SU(2) are thus realised as the symmetry group acting on $q_m S(E_n; m)$, up to equivalence. In particular, in the case when $n = |m|$, the symmetry group is better considered as

(i) SO(3) if m is even

(ii) SU(2) if m is odd.

7. Generators of the symmetry group

In this section we construct the generators of the symmetry group found in § 6. The reduction method used in § 4 is also applicable in this section.

We recall that the Hamiltonian $\hat{H}$ of the quantised conformal Kepler problem was reduced to the operator $\hat{H}_m$ because of the invariance $U_t\hat{H}U_t^{-1}=\hat{H}$. The reduction is valid for other U(1)-invariant operators.

Lemma 7.1. Let $\hat{F}$ be an operator defined in $L^2(\boldsymbol{R}^4; 4r\,\mathrm{d}x)$. Then $\hat{F}$ can be reduced to an operator defined in Γ_m for any m, if and only if $\hat{F}$ is U(1) invariant:

$$U_t\hat{F}U_t^{-1}=\hat{F}. \tag{62}$$

Proof. We restrict the domain of $\hat{F}$ to, say, $C_0^\infty(\boldsymbol{R}^4)$. Let $\hat{F}$ be U(1) invariant. Then for ρ_m-equivariant functions f, one can define a reduced operator $\hat{F}_m$ through

$$\hat{F}_m(q_m f)=q_m(\hat{F}f) \tag{63}$$

because $\hat{F}f$ is also ρ_m equivariant;

$$U_t\hat{F}f=\hat{F}U_tf=\exp(-\mathrm{i}mt/2)\hat{F}f. \tag{64}$$

Conversely, if $\hat{F}$ is reduced to $\hat{F}_m$, satisfying (63), then $\hat{F}f$ must be ρ_m equivariant, so that $U_t\hat{F}f=\exp(-\mathrm{i}mt/2)\hat{F}f=\hat{F}U_tf$. Thus, one finds that $\hat{F}$ is U(1) invariant on the space of ρ_m-equivariant functions. We now wish to show that $\hat{F}$ is also U(1) invariant on $C_0^\infty(\boldsymbol{R}^4)$.

Let f be any function in $C_0^\infty(\boldsymbol{R}^4)$. Then the function f_m defined by

$$f_m(x)=\frac{1}{4\pi}\int_0^{4\pi} f(T(t)x)\exp(-\mathrm{i}mt/2)\,\mathrm{d}t \tag{65}$$

is ρ_m equivariant, because

$$\begin{aligned} f_m(T(s)x)&=\frac{1}{4\pi}\int_0^{4\pi} f(T(t)T(s)x)\exp(-\mathrm{i}mt/2)\,\mathrm{d}t\\ &=\frac{1}{4\pi}\int_0^{4\pi} f(T(s+t)x)\exp[-\mathrm{i}m(s+t)/2]\exp(\mathrm{i}ms/2)\,\mathrm{d}(s+t)\\ &=\exp(\mathrm{i}ms/2)f_m(x). \end{aligned} \tag{66}$$

Since $f_m(x)$ are Fourier coefficients, $f(T(t)x)$ is expanded into

$$f(T(t)x)=\sum_{m=-\infty}^{\infty} f_m(x)\exp(\mathrm{i}mt/2). \tag{67}$$

From this it follows that any function is expanded as a sum of ρ_m-equivariant functions, where m varies over all the integers. Accordingly, $\hat{F}$ becomes U(1) invariant on $C_0^\infty(\boldsymbol{R}^4)$, as a consequence of its U(1) invariance on any space of ρ_m-equivariant functions. This completes the proof.

What we have to do now is to find U(1)-invariant operators which generate the symmetry subgroup $\mathrm{SU}(2)\times\mathrm{SU}(2)$ for the quantised conformal Kepler problem. Incidentally, a method for obtaining such generators is given in Iwai (1982a), according to which the following is a basis of U(1)-invariant operators which generate $\mathrm{SU}(2)\times\mathrm{SU}(2)$:

$$
\begin{aligned}
\hat{J}_1 &= \frac{1}{2\mathrm{i}}\left(x_1\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\right)\\
\hat{J}_2 &= \frac{1}{2\mathrm{i}}\left(x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_2}\right)\\
\hat{J}_3 &= \frac{1}{2\mathrm{i}}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right)\\
\hat{D}_1 &= -\frac{1}{4}\left(\frac{\partial^2}{\partial x_1\,\partial x_3} + \frac{\partial^2}{\partial x_2\,\partial x_4}\right) - 2(x_1x_3 + x_2x_4)\hat{H}\\
\hat{D}_2 &= -\frac{1}{4}\left(\frac{\partial^2}{\partial x_2\,\partial x_3} - \frac{\partial^2}{\partial x_1\,\partial x_4}\right) - 2(x_2x_3 - x_1x_4)\hat{H}\\
\hat{D}_3 &= -\frac{1}{8}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}\right) - (x_1^2 + x_2^2 - x_3^2 - x_4^2)\hat{H}
\end{aligned}
\tag{68}
$$

where $\hat{H}$ is the Hamiltonian operator of the quantised conformal Kepler problem. These are symmetric operators on $C_0^\infty(\boldsymbol{R}^4)$ with respect to the inner product (3) (see Iwai 1982a). The commutation relations among $\hat{J}$ and $\hat{D}$ are given by

$$
[\hat{J}_j, \hat{J}_k] = \mathrm{i}\varepsilon_{jkh}\hat{J}_h \qquad [\hat{D}_j, \hat{D}_k] = \mathrm{i}\varepsilon_{jkh}\hat{J}_h(-2\hat{H}) \qquad [\hat{J}_j, \hat{D}_k] = \mathrm{i}\varepsilon_{jkh}\hat{D}_h \tag{69}
$$

where ε_{jkh} are Edington's epsilon and j, k and h range over 1, 2, 3. We are now in a final stage to get the symmetry operators. As we have done in (39) for $\hat{H}$, we express the operators (68) in terms of $(\partial/\partial\tilde{x}_j)^*$ and $\hat{N}$. A calculation results in

$$
\begin{aligned}
\hat{J}_1 &= \frac{1}{\mathrm{i}}\left[\tilde{x}_2\left(\frac{\partial}{\partial\tilde{x}_3}\right)^* - \tilde{x}_3\left(\frac{\partial}{\partial\tilde{x}_2}\right)^*\right] + \frac{\tilde{x}_1}{r}\hat{N}\\
\hat{J}_2 &= \frac{1}{\mathrm{i}}\left[\tilde{x}_3\left(\frac{\partial}{\partial\tilde{x}_1}\right)^* - \tilde{x}_1\left(\frac{\partial}{\partial\tilde{x}_3}\right)^*\right] + \frac{\tilde{x}_2}{r}\hat{N}\\
\hat{J}_3 &= \frac{1}{\mathrm{i}}\left[\tilde{x}_1\left(\frac{\partial}{\partial\tilde{x}_2}\right)^* - \tilde{x}_2\left(\frac{\partial}{\partial\tilde{x}_1}\right)^*\right] + \frac{\tilde{x}_3}{r}\hat{N}\\
\hat{D}_1 &= \frac{1}{2\mathrm{i}}\left[\hat{J}_2\left(\frac{\partial}{\partial\tilde{x}_3}\right)^* - \hat{J}_3\left(\frac{\partial}{\partial\tilde{x}_2}\right)^* - \left(\frac{\partial}{\partial\tilde{x}_2}\right)^*\hat{J}_3 + \left(\frac{\partial}{\partial\tilde{x}_3}\right)^*\hat{J}_2\right] + k\frac{\tilde{x}_1}{r}\\
\hat{D}_2 &= \frac{1}{2\mathrm{i}}\left[\hat{J}_3\left(\frac{\partial}{\partial\tilde{x}_1}\right)^* - \hat{J}_1\left(\frac{\partial}{\partial\tilde{x}_3}\right)^* - \left(\frac{\partial}{\partial\tilde{x}_3}\right)^*\hat{J}_1 + \left(\frac{\partial}{\partial\tilde{x}_1}\right)^*\hat{J}_3\right] + k\frac{\tilde{x}_2}{r}\\
\hat{D}_3 &= \frac{1}{2\mathrm{i}}\left[\hat{J}_1\left(\frac{\partial}{\partial\tilde{x}_2}\right)^* - \hat{J}_2\left(\frac{\partial}{\partial\tilde{x}_1}\right)^* - \left(\frac{\partial}{\partial\tilde{x}_1}\right)^*\hat{J}_2 + \left(\frac{\partial}{\partial\tilde{x}_2}\right)^*\hat{J}_1\right] + k\frac{\tilde{x}_3}{r}.
\end{aligned}
\tag{70}
$$

Operating ρ_m-equivariant functions with (70) and applying (21), we obtain the reduced operators

$$
\begin{aligned}
[\hat{J}_1]_m &= (1/\mathrm{i})(\tilde{x}_2\nabla_3 - \tilde{x}_3\nabla_2) + [(m/2)/r]\tilde{x}_1\\
[\hat{J}_2]_m &= (1/\mathrm{i})(\tilde{x}_3\nabla_1 - \tilde{x}_1\nabla_3) + [(m/2)/r]\tilde{x}_2
\end{aligned}
$$

$$
\begin{aligned}
&[\hat{J}_3]_m=(1/\mathrm{i})(\tilde{x}_1\nabla_2-\tilde{x}_2\nabla_1)+[(m/2)/r]\tilde{x}_3\\
&[\hat{D}_1]_m=(1/2\mathrm{i})([\hat{J}_2]_m\nabla_3-[\hat{J}_3]_m\nabla_2-\nabla_2[\hat{J}_3]_m+\nabla_3[\hat{J}_2]_m)+k(\tilde{x}_1/r)\\
&[\hat{D}_2]_m=(1/2\mathrm{i})([\hat{J}_3]_m\nabla_1-[\hat{J}_1]_m\nabla_3-\nabla_3[\hat{J}_1]_m+\nabla_1[\hat{J}_3]_m)+k(\tilde{x}_2/r)\\
&[\hat{D}_3]_m=(1/2\mathrm{i})([\hat{J}_1]_m\nabla_2-[\hat{J}_2]_m\nabla_1-\nabla_1[\hat{J}_2]_m+\nabla_2[\hat{J}_1]_m)+k(\tilde{x}_3/r).
\end{aligned}
\tag{71}
$$

In conclusion we make a mention of quantisation. In Iwai and Uwano (1986), constants of motion for the reduced classical system are defined on the cotangent bundle of $\dot{\boldsymbol{R}}^3$. If we make it a quantisation rule to replace $\tilde{p}_j$ by $-\mathrm{i}\nabla_j$, we can obtain (71) from the classical constants of motion. If $m=0$, the operators (71) reduce to the angular momentum and Runge-Lenz vector for the hydrogen atom. The operators (71) were given in McIntosh and Cisneros (1970) in terms of vector potential. Our result is, however, expressed in the large. Note also that the angular momentum operators $[\hat{J}]_m$ are treated in Wu and Yang (1976) for discussing the monopole harmonics.

This section is summed up in the following.

Theorem 7.2. The U(1)-invariant operators (68) which generate the symmetry subgroup SU(2) × SU(2) for the quantised conformal Kepler problem are reduced to the symmetry operators (71) for the quantised MIC-Kepler problem $(\Gamma_m, \hat{H}_m)$. The $[\hat{J}]_m$ are the total angular momentum and the $[\hat{D}]_m$ the Runge-Lenz-like vector.

8. Concluding remarks

To understand in a conceptual manner how the monopole field arises in the reduced system, we have to express the reduction of the Hamiltonian operator in terms of curvilinear coordinates which are related to the Euler angles. Let us define the curvilinear coordinates in $\dot{\boldsymbol{R}}^4$ by

$$
\begin{aligned}
&x_1=\sqrt{r}\cos(\theta/2)\cos[(\psi+\phi)/2] && x_2=\sqrt{r}\cos(\theta/2)\sin[(\psi+\phi)/2]\\
&x_3=\sqrt{r}\sin(\theta/2)\cos[(\psi-\phi)/2] && x_4=\sqrt{r}\sin(\theta/2)\sin[(\psi-\phi)/2]
\end{aligned}
\tag{72}
$$

where $r=\Sigma_{j=1}^{j=4}x_j^2$ and the variables (θ,ϕ,ψ) range over $0\leqslant\theta\leqslant\pi$, $0\leqslant(\psi+\phi)/2\leqslant 2\pi$ and $-\pi\leqslant(\psi-\phi)/2\leqslant\pi$, respectively.

Then the quantised conformal Kepler Hamiltonian operator is written as

$$
\begin{aligned}
\hat{H}=-\frac{1}{2}\bigg[&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}\right)\\
&+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial}{\partial\phi}-\cos\theta\frac{\partial}{\partial\psi}\right)^2+\frac{1}{r^2}\frac{\partial^2}{\partial\psi^2}\bigg]-\frac{k}{r}
\end{aligned}
\tag{73a}
$$

and the angular momentum operator as

$$\hat{N}=\mathrm{i}\partial/\partial\psi. \tag{73b}$$

Let us recall the discussion on the derivation of the vector potentials for the monopole field (equations (29)-(35)). One easily verifies, from (29) and (30) and definition (72), that the ρ_m-equivariant function $\chi_+(x)$ given in (29) is equal to

$\exp[\mathrm{i}m(\psi+\phi)/2]$, so that the ρ_m-equivariant function $f(x)$ on $\dot{\boldsymbol{R}}^4$ which corresponds to an arbitrary local cross section $f^+(r, \theta, \phi)\gamma_+$ in L_m can be written as

$$f(x)=f(r, \theta, \phi, \psi)=f^+(r, \theta, \phi)\exp[\mathrm{i}m(\psi+\phi)/2] \tag{74}$$

on $\pi^{-1}(U_+)=\{(r, \theta, \phi, \psi);\ \theta\neq\pi\}$. Operating this $f(x)$ with the Hamiltonian $\hat{H}$, we have

$$\begin{aligned}(\hat{H}f)(r, \theta, \phi, \psi) &= \hat{H}(f^+\exp[\mathrm{i}m(\psi+\phi)/2])\\ &=\left\{-\frac{1}{2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}\right)\right.\right.\\ &\quad\left.+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial}{\partial\phi}+\mathrm{i}\frac{m}{2}(1-\cos\theta)\right)^2\right]\\ &\quad\left.+\frac{(m/2)^2}{2r^2}-\frac{k}{r}\right\}f^+\exp[\mathrm{i}m(\psi+\phi)/2].\end{aligned} \tag{75}$$

Thus, from (38), we have

$$\begin{aligned}\hat{H}_m = &-\frac{1}{2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}\right)\right.\\ &\left.+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial}{\partial\phi}+\mathrm{i}\frac{m}{2}(1-\cos\theta)\right)^2\right]+\frac{(m/2)^2}{2r^2}+\frac{k}{r}\end{aligned} \tag{76}$$

which gives a local expression to the Hamiltonian operator $\hat{H}_m$ for the quantised MIC-Kepler problem on U_+. Compare (76) with (41).

The local expression (76) of the reduced Hamiltonian shows that the reduced system is coupled with the magnetic field $\mathrm{i}(m/2)\Omega=\mathrm{i}(m/2)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$ whose vector potential on U_+ is $\mathrm{i}(m/2)(1-\cos\theta)\,\mathrm{d}\phi$. Recall that any equivariant function is an eigenfunction of $\hat{N}$ (see (18)). Then, we observe that the angular momentum possessed by the original system exerts an influence on the reduced system through the magnetic field $\mathrm{i}(m/2)\Omega$. The centrifugal potential $(m/2)^2/2r^2$ is another influence.

Note that the derivation of (76) is valid only on U_+ because of singularities of the vector potential for the monopole field. This is the reason why we introduce the complex line bundles L_m in order to analyse globally the quantised MIC-Kepler problem.

We here refer to the work of Iwai (1987) on the reduction of the quantum planar three-body problem by an S^1 action, in which one can understand clearly the appearance of the monopole field induced by the rotation. According to Iwai (1987), the centre-of-mass system is made into a principal SO(2) bundle $\dot{\boldsymbol{R}}^4\to\dot{\boldsymbol{R}}^3$ and the SO(2) action is given as the rotation around the centre of mass for the three bodies. As in our case, the resultant quantum-reduced system carries the monopole field and the centrifugal potential. In his case, the monopole field can be interpreted as the Coriolis field induced by the rotation.

We point out, in addition, that the monopole field or a monopole-like field can be found in the literature concerning the SO(2) reduction for classical Hamiltonian systems (Satzer 1977, Kummer 1981, Marsden 1981). For instance, the planar n-harmonic oscillator system is reduced, by an SO(2) action, to the system having the monopole-like field and the centrifugal potential, which was pointed out by Weinstein (Marsden 1981). The monopole-like field is also understood as the 'magnetic' field induced by the rotation.

To close this section, we mention a relation between monopole harmonics (Wu and Yang 1976) and the eigensections for the quantised MIC-Kepler problem. To do this, we express the quantised conformal Kepler problem in the form

$$\hat{H} = -\frac{1}{2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{4r^2}\Delta_3\right] - \frac{k}{r} \tag{77}$$

where Δ_3 is the standard Laplacian on the unit sphere $S^3 \subset \boldsymbol{R}^4$. If we follow the procedure (14)-(16) with S^3 and S^2 in place of $\dot{\boldsymbol{R}}^4$ and $\dot{\boldsymbol{R}}^3$, respectively, we have the complex line bundle $S^3 \times_m \boldsymbol{C} \to S^2$. Further, in the same manner as (38), the Laplacian Δ_3 on S^3 is reduced to the operator $\Delta_2^{(m)} - m^2$ acting on cross sections in $S^3 \times_m \boldsymbol{C}$ (see Kuwabara (1982) with different notation). Here $\Delta_2^{(m)}$ is called the Bochner Laplacian. According to Wu and Yang (1976) and Dray (1986), monopole harmonics are defined as eigen cross sections for $\Delta_2^{(m)} - m^2$ in the complex line bundle $S^3 \times_m \boldsymbol{C}$. We would instead like to define monopole harmonics as eigen cross sections for $\Delta_2^{(m)}$. Since the eigenvalues of Δ_3 are $-4J(J+1)$ with J non-negative half integers, one has for the monopole harmonics σ,

$$\Delta_2^{(m)}\sigma = -[4J(J+1) - m^2]\sigma$$

with $J = |m|/2,\ |m|/2+1, |m|/2+2, \ldots$. Using $\Delta_2^{(m)}$, we can express the quantised MIC-Kepler Hamiltonian $\hat{H}_m$ in the form

$$\hat{H}_m = -\frac{1}{2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{4r^2}\Delta_2^{(m)}\right] + \frac{(m/2)^2}{2r^2} - \frac{k}{r}. \tag{78}$$

In view of this, we can take an ansatz $R(r)\sigma$ as an eigen cross section for $\hat{H}_m$, where σ is the monopole harmonics.

We turn to a local expression of the eigen cross section for $\hat{H}_m$. To this end, we first give a local expression to the eigenfunctions of the quantised conformal Kepler problem of eigenvalue E_n in terms of the curvilinear coordinates (r, θ, ϕ, ψ) in (72). Using the expression (73*a*), we obtain, after calculation, the eigenfunctions (not normalised) for $\hat{H}$ of eigenvalue $E_n = -2k^2/(n+2)^2$ (with n a non-negative integer)

$$F_{K,M}^{(n,J)}(r, \theta, \phi, \psi) := R_J^n(r) Q_{K,M}^J(\theta, \phi, \psi) \tag{79a}$$

with

$$\begin{aligned}
&R_J^n(r) = \exp(-\varepsilon_n r/2)(\varepsilon_n r)^J L_{J+1+n/2}^{2J+1}(\varepsilon_n r)\\
&Q_{K,M}^J(\theta, \phi, \psi) = \mathrm{e}^{\mathrm{i}K\phi}\,\mathrm{e}^{\mathrm{i}M\psi} P_\gamma^{(\alpha,\beta)}(\cos\theta)\\
&\varepsilon_n = \frac{4k}{n+2}\\
&\alpha = |K-M| \qquad \beta = |K+M| \qquad \gamma = J - \tfrac{1}{2}(|K-M| + |K+M|).
\end{aligned} \tag{79b}$$

Here, J, K, and M are half-integers subject to

$$0 \leqslant (n/2) - J \in \boldsymbol{Z} \qquad 0 \leqslant J - |K| \qquad J - |M| \in \boldsymbol{Z} \tag{79c}$$

with $\boldsymbol{Z}$ denoting the set of all integers. The functions $L_{J+1+n/2}^{2J+1}$ and $P_\gamma^{(\alpha,\beta)}$ are, respectively, the associated Laguerre polynomial and the Jacobi polynomial (for definition, see Erdelyi *et al* (1953)). The expression (79) is related to that obtained for the harmonic oscillator (Ikeda and Miyachi 1970) through (6). The $F^{(n,J)}_{K,M}$ become ρ_m equivariant iff $M = m/2$, since one has, by calculation,

$$U_t F^{(n,J)}_{K,M}(r, \theta, \phi, \psi) = F^{(n,J)}_{K,M}(r, \theta, \phi, \psi - t)$$
$$= e^{-iMt} F^{(n,J)}_{K,M}(r, \theta, \phi, \psi) \qquad (t \in [0, 4\pi]). \tag{80}$$

Thus from (74) we have a local expression of the eigen cross section for $\hat{H}_m$ as

$$f^+(r, \theta, \phi) = R^n_J(r) \exp[i(K - m/2)\phi] P^{(\alpha,\beta)}_\gamma(\cos\theta)$$

with

$$\alpha = |K - m/2| \qquad \beta = |K + m/2| \qquad \text{and} \qquad \gamma = J - \tfrac{1}{2}(\alpha + \beta).$$

The spherical part of this f^+ gives a local expression of the monopole harmonics.

It should be pointed out that the spherical part $Q^J_{K,M}$ of $F^{(n,J)}_{K,M}$, which is an eigenfunction for Δ_3, gives nothing but the Wigner $\mathscr{D}$ function, $\mathscr{D}^J_{K,M}$ of weight J (for a definition, see Edmonds (1960)) up to a multiplicative constant. According to Dray (1986), the $Q^J_{K,m/2}$ give rise to monopole harmonics of monopole strength $-m/2$. In our notation, the local cross section, $q_m(Q^J_{K,m/2})$, of the complex line bundle L_m is the very monopole harmonics of strength $-m/2$.

Wu and Yang (1976) discussed completely how to piece together locally defined monopole harmonics into global ones, i.e. cross sections. Using $q_m(Q^J_{K,m/2})$, Greub and Petry (1975) obtained the eigenfunctions for a particle moving in the magnetic monopole field. In Kuwabara (1982), the same functions, $\{Q^J_{K,M}\}$, are utilised to give the eigen cross sections for the Bochner Laplacian $\Delta^{(m)}_2$ of the complex line bundle, $S^3 \times_m \boldsymbol{C} \to S^2$, over S^2.

Appendix

This appendix shows that $U^{(n)}_C$ acts unitarily on $S(E_n)$ with respect to the inner product $\langle\ ,\ \rangle$ defined by (3).

First, we show the orthogonality of the basis functions of $S(E_n)$, $\{\psi_{\boldsymbol{l}}\}_{l_1+l_2+l_3+l_4=n}$ given by (9) with $\lambda = 4k/(n+2)$. Let us define a function $h_j(x)$ by

$$h_j(x) = (\lambda/\pi)^{1/4}(2^j j!)^{-1/2} H_j(\sqrt{\lambda}x)\exp(-\lambda x^2/2) \qquad (\lambda > 0)$$

where $H_j(x)$ is the Hermite polynomial (see Erdelyi *et al* 1960):

$$H_j(x) = (-1)^j \exp(x^2)(\mathrm{d}/\mathrm{d}x)^j \exp(-x^2).$$

Then $\psi_{\boldsymbol{l}} \in S(E_n)$ is written in the form

$$\psi_{\boldsymbol{l}}(x) = h_{l_1}(x_1)h_{l_2}(x_2)h_{l_3}(x_3)h_{l_4}(x_4)$$

where the bold subscript $\boldsymbol{l}$ is a multi-index denoting $(l_1l_2l_3l_4)$. Let $(\ ,\ )$ denote the standard inner product in $L^2(\boldsymbol{R})$. Then because of $r = \sum_{k=1}^{k=4} x_k^2$, the inner product of $\psi_{\boldsymbol{i}}$ and $\psi_{\boldsymbol{j}}$ takes the form

$$\langle \psi_{\boldsymbol{i}}, \psi_{\boldsymbol{j}} \rangle = 4\sum_{m=1}^{4} [(h_{i_1}, h_{j_1}) \ldots \widehat{(h_{i_m}, h_{j_m})} \ldots (h_{i_4}, h_{j_4})](h_{i_m}, x_m^2 h_{j_m}). \tag{A1}$$

Here, the caret means that the factor (h_{i_m}, h_{j_m}) in the square brackets is excluded. Since the relation $(h_i, h_j) = \delta_{ij}$ is easily verified (cf Erdelyi *et al* 1953), we have only to examine (h_i, x^2h_j) for the orthogonality of ψ_j with respect to $\langle\ ,\ \rangle$. A straightforward calculation shows that, for $i \leqslant j$,

$$(h_i, x^2h_j) = \begin{cases} (2j+1)/2\lambda & i=j \\ [j(j-1)]^{1/2}/2\lambda & i+2=j \\ 0 & \text{otherwise.} \end{cases} \tag{A2}$$

The assumption $i \leqslant j$ does not cause the loss of generality of (A2) because $(h_i, x^2h_j) = (x^2h_i, h_j)$ holds. When ψ_i and ψ_j belong to $S(E_n)$, multi-indices $\boldsymbol{i}$ and $\boldsymbol{j}$ satisfy

$$i_1+i_2+i_3+i_4 = j_1+j_2+j_3+j_4 = n.$$

Under this condition, possible choices of $\boldsymbol{i}$ and $\boldsymbol{j}$ are divided into the following: (a) $\boldsymbol{i}=\boldsymbol{j}$, ($b$) $i_m = j_m$ for two m, (c) $i_m = j_m$ for only one m, (d) $i_m \neq j_m$ for any m. In the case of (a), the content of the square brackets in (A1) always equals unity, while it vanishes in the cases of (b), (c), and (d). From (A1) and (A2) we obtain, in $S(E_n)$,

$$\langle \psi_i, \psi_j \rangle = 4(n+2)\delta^i_j/\lambda = \frac{(n+2)^2}{k}\delta^i_j \tag{A3}$$

setting $\lambda = 4k/(n+2)$ (see § 2).

We are now in a position to show the unitarity of the $U^{(n)}_C$ action on $S(E_n)$. Since $U^{(n)}_C$ can be regarded as a finite-dimensional linear transformation in $S(E_n)$, $U^{(n)}_C\psi_i$ can be expressed as

$$(U^{(n)}_C\psi_j)(x) = \sum_i A^i_j\psi_i(x) \qquad (A^i_j \in \boldsymbol{C}). \tag{A4}$$

As $U^{(n)}_C$ is a unitary action with respect to the standard inner product in $L^2(\boldsymbol{R}^4)$, the matrix (A^i_j) is a unitary matrix:

$$\sum_m \overline{A^m_i}A^m_j = \delta^i_j. \tag{A5}$$

Using (A3)-(A5), we have, in $S(E_n)$,

$$\langle U^{(n)}_C\psi_i, U^{(n)}_C\psi_j \rangle = \sum_{h,m} \overline{A^h_i}A^m_j\langle \psi_h, \psi_m \rangle = \frac{(n+2)^2}{k}\sum_m \overline{A^m_i}A^m_j = \frac{(n+2)^2}{k}\delta^i_j = \langle \psi_i, \psi_j \rangle.$$

This proves our assertion.

References

Chen A C and Kibler M 1985 *Phys. Rev.* A **31** 3960-3

Crampin M 1981 *J. Phys. A: Math. Gen.* **14** 3407-9

Dray T 1986 *J. Math. Phys.* **27** 781-92

Edmonds 1960 *Angular Momentum in Quantum Mechanics* (Princeton, NJ: Princeton University Press)

Erdelyi A, Magnus W, Oberhettinger F and Tricomi G 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill)

Greub W and Petry H-R 1975 *J. Math. Phys.* **16** 1347-51

Ikeda M and Miyachi Y 1970 *Math. Japan* **15** 127-42

Iwai T 1981 *J. Math. Phys.* **22** 1633-9

—— 1982a *J. Math. Phys.* **23** 1093-9

—— 1982b *J. Math. Phys.* **23** 1088-92

—— 1987 *J. Math. Phys.* **28** 964-74

Iwai T and Uwano Y 1986 *J. Math. Phys.* **27** 1523-9
Kibler M and Negadi T 1983 *Lett. Nuovo Cimento* **37** 225-8
—— 1984a *Lett. Nuovo Cimento* **39** 319-23
—— 1984b *Croatia Chem. Acta* **57** 1509-23
Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 (New York: Wiley) ch 2
Kummer M 1981 *Indiana Univ. Math. J.* **30** 281-91
Kuwabara R 1982 *J. Math. Tokushima Univ.* **16** 1-23
Marsden J E 1981 *Lectures on Geometric Methods in Mathematical Physics* (Philadelphia, PA: SIAM) lecture 4
McIntosh H V and Cisneros A 1970 *J. Math. Phys.* **11** 896-916
Ryder L H 1980 *J. Phys. A: Math. Gen.* **13** 437-47
Satzer W J Jr 1977 *Indiana Univ. Math. J.* **26** 951-76
Schonfeld J F 1980 *J. Math. Phys.* **21** 2528-33
Wigner E P 1959 *Group Theory* (New York: Academic) ch 16
Wu T T and Yang C N 1976 *Nucl. Phys.* B **107** 365-80